**Interaction**

Now let’s allow the lattice to move. Then, combining the electron-electron interaction and the electron-phonon interaction, we get:



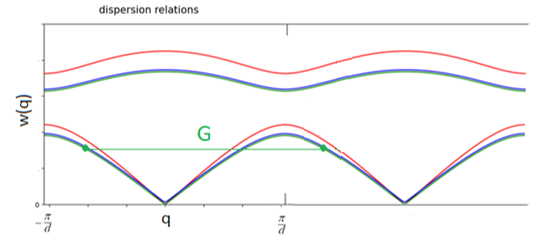
where,



where nion is number density, and ρion is mass density. Rules are perhaps fairly self-evident. But for the sake of completeness…

\* note I took out the q ≠ 0 restriction in the e-e interaction term, as that happened because we approximated the ionic lattice as a positive uniform background of charge. But we’re not anymore, so we shouldn’t have that restriction anymore. Right? We do have that restriction on the electron-phonon term though, because q = 0 corresponds to uniform displacement of ions, which isn’t under consideration here.

And don’t forget that in the electron-phonon interaction, sums over phonon wavevectors extend over all q’s, but Ωqλ, and Aqλ (and thereby the phonon GF too) are periodic functions of q. Looks like this, illustrated for an acoustic spectrum of phonons, which we haven’t demonstrated yet, but you get the idea.



Okay, moving on to the…

**Feynman Diagram Rules**

Say we’re interested in calculating the (complex time) single particle GF for electrons,



or phonons,



This is of course,



where,



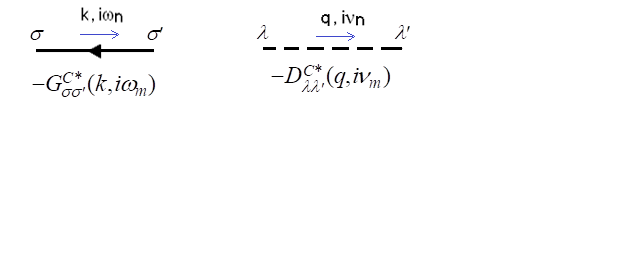
and,



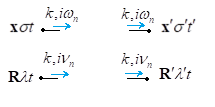
The Feynman rules for single particle GF’s are, basically without any justification…

**Fourier Space Rules**

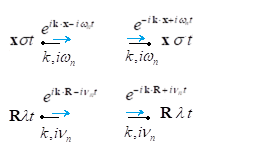
So we have our single GF’s:



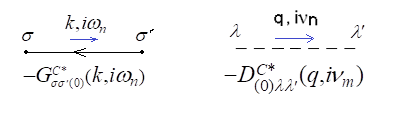
and the external points:



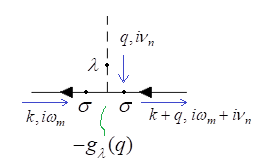
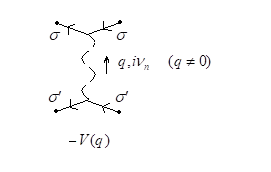
These carry the Fourier transform variable. And if we’re ultimately interested in the real space, time GF, then we must add the following factors to each:



and we connect these with the bare GF’s



And we associate the Veph and Vee interactions with:

(well, we only associate -gλ(q) with Veph, but I was illustrating the energy/momentum conservation that takes place at the vertex). Also, if q = 0 on the phonon line, then we get zero, like happens with the e-e interaction\*. And apropos Vee we still have iνn flowing down the line because of the artificial time-dependence brought in by the δ(t-t´) we put on each interaction potential. Note it’s a bosonic frequency, though, as energy conservation demands, because we cannot have electron lines with fermionic frequencies if we have a fermionic interaction frequency traveling into it (EM interaction is carried by bosons too – is this a coincidence?).

\* not sure we shouldn’t have q = 0 for e-e interaction term, ‘cause we’re not working with jelly background anymore.

**Topology**

Connect all topologically distinct, fully connected (meaning no vacuum bubbles), diagrams together, associating with each element in the diagram the indicated term. Remember that all energy-momentum labels **k**, ω must be going in the same way w/r to the GF arrow. They must all be going against the arrows (or with them I suppose). And then we conserve energy/momentum at each vertex.

**Equal time issues**

Apropos the equal time issue, in Fourier space this would be handled with the customary exp(iωn0+) for each bare Green’s function closed in on itself. Fundamentally, the problem occurs when the temporal argument of the bare Green’s function is zero.

**Signs/Numerical Factors**

Every diagram gets a factor of 1 I believe. (Fermion) loops have their own special sign factors. Each fermion loop get a factor of (-ε), i.e., -1. Vacuum bubbles, should we need to calculate them (perhaps when calculating thermodynamic potentials) have their own symmetry factors. I’m not sure what the phonon ones are. Can look up the e-e ones in that ee file. Each Fermion loop does get a factor of -ε of course.

**Sum**

Then sum/integrate over all independent momenta - wavenumbers/energies-frequencies/indices (and spins, polarizations).



Might observe that the point of factoring out that 1/√V in Veph was to make sure that there were two such factors, and hence a net factor of 1/V for every appearance of the phonon GF, so the customary (1/V)Σk rule above would be preserved. Also, since the unperturbed GF’s are presumably diagonal in σσ´, and λλ´, the internal sum over spins and polarizations will just cancel δ functions, with the net effect that whatever spin/polarization we put it in will come out….except when have fermion loops. Then we need to multiply by (2s+1) = 2, as we’ve discussed before. Also, for a homogeneous medium, we’ll note that the interaction, gλ(q) = g(q)δλ3, and so when we do the sum over polarizations, the gλ(q) will just pull out G33 alone. The transverse GF’s won’t be coupled to the interaction. So all transverse polarization GF’s won’t even be affected. So basically, we can just ignore them and pretend like D33C\* is our only guy.

(Fermion) loops have there own special sign factors. Each fermion loop get a factor of (-ε), which would be just 1 for a boson loop – this has to do with intracacies of transposing the operators to put them in proper order.

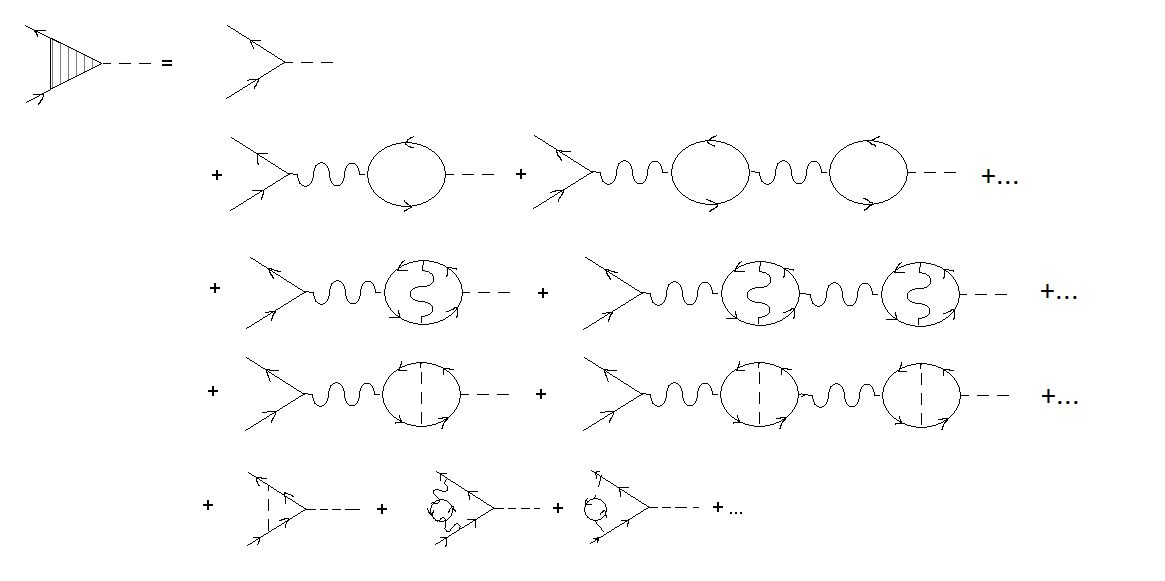
**Effective interactions**

Say we’re interested in studying the behavior of phonons…in the presence of interacting electrons, or that we’re interested in concentrating on the behavior of electrons…in the presence of the phonons. Then we might seek for an ‘effective’ interaction between e-ph on the one hand, or e-e on the other.

What follows would be more formally justified, I think, in the path-integral language. Like has been done in the path integral formulations of GF’s in previous folders, we’d set up, say, an expression for Ξ. This would involve a functional integration over the electron states (Grassman numbers) and phonon states (complex numbers). And to get an effective interaction between the electrons alone, we could peform the integration over the phonons, leaving us with just an expression for the Ξ in terms of electron d.o.f., and their ‘effective interaction’. This is like what we do in the Stat Mech folder, in the RG file, where we integranded over small wavelength modes to get an effective H for long wavelenths. We could do analogously if we wanted to know the ‘effective’ interaction between phonons. Anyway, I imagine the effective interactions which result from such a procedure is equivalent to the diagramatic approach we’re going to take below. The advantage of the path integral approach is that it isn’t as ad hoc as the kind of blind diagrammatic approach is.

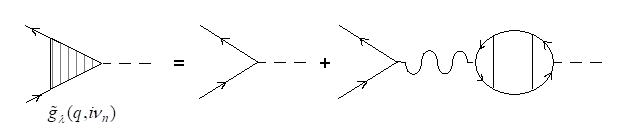
**electron-phonon interaction**

So the bare electron-phonon interaction is a process whereby an electron scatters into another state by absorbing/emitting a phonon. We can generalize this process to include any interaction whereby an electron starts in one state and ends in another with the net result being that a phonon has been absorbed/emitted. If we add all of these processes together, we’d get an ‘effective’ electron-phonon interaction, illustrated below:



We may interpret this as an electron absorbing/emitting a phonon in the first instance. The second would be an electron scattering off another, the energy of which creates creates an electron/anti-electron pair (virtual) which recombine to release a phonon, etc.

Apropos the last line, Migdal’s theorem states that the phonon corrections to the vertex are of order √(m/M) ~ 10-2 relative to the bare vertex g(q) and hence may be neglected. Therefore we simply have the top four lines which can be expressed as,



where is the effective electron-phonon vertex, and

Shape

Description automatically generated

is the two-particle correlation function (note, not irreducible). And in terms of this, we can write, for the effective interaction,



and so,



We can work on an approximation to this. The most important contributions to Π are the purely e-e guys. Adding all these up is the RPA approximation. So I guess ultimately we’re just considering screening of the e-ph interaction by other electrons. One usually neglects the frequency dependence of . I guess we’re just saying that in the scheme of things, the ionic oscillations aren’t that important vis a vis their interactions with each other? I don’t know. Then, we have:



So we’re basically replacing the bare electron-ion potential (basically a Coulomb potential), with the screened electron-ion potential. We can make one more kind of cosmetic change at this point. As we’ll see in the Excitations folder, the same electron screening changes the phonon Green’s function’s poles to:



And the GF itself to:



Now note that in any diagram, for instance,



two vertices will be connected to a single GF. So they always come in the combination λ2D, so we can make the replacements,



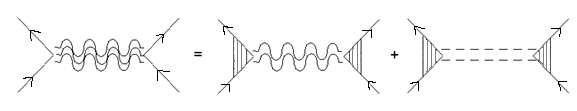
Thus the renormalized phonon GF takes the unperturbed form, but with the renormalized frequencies. And the renormalized (twice) interaction vertex also takes the unperturbed form, but with a screened potential, and renormalized frequencies. These equations can be taken as the starting point for a treatment of phonons that otherwise ignores the effects of the Coulomb interactions. Maybe I’ll work out the new g a little,



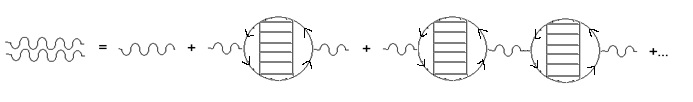
So in the small q limit, we can see it’s appropriate to treat the screened potential as a delta function.

**electron-electron interaction**

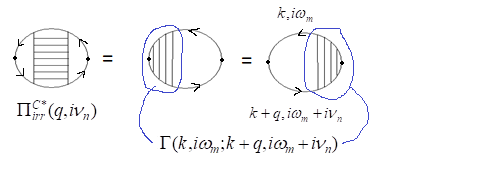
Now consider the effective e-e interaction. The effective interaction describes any generic process whereby two incoming electrons scatter off each other into different outgoing states (I’m thinking that if we had a sort of partition function for this system, and we summed over all phonon states, we’d end up with a partition function purely in terms of electron states, and this effective e-e interaction we’re discussing is what would show up in the effective Hamiltonian). We can write it generically as the sum of two contributions (leaving out some possible diagrams though):



Note these vertex corrections are the same as the ones above in the e-ph interaction. The double straight line (also bold line above) is the exact phonon GF. The double wavy line is:



where that guy is the irreducible two-point function, Π­irrC\*(q,iνn), which can also be written in terms of a vertex function, Γ(k,iωm;k+q,iωm+iνn) (momenta/energy going against arrows)



Written out, we have:



But he calls this simply,



Looks like this is just a definition, because Γ does not decouple. Sometimes we may presume Γ doesn’t depend on k or iωm, in which case it could be identified with Λ(q,iν­n). Anyway, we’ll note that Γ (or Λ) is the same term as resides on the vertices of the interaction. So altogether, we’d have:



If we drop the vertex corrections, which would be equivalent to using the RPA approximation of ΠirrC\*(q,iνn), we’d just have the more familiar:



Now let’s consider the second diagram in (need to use results of phonon excitations file for DC\*) Borrowing,



we have (using the original renormalized vertex),



where we use, from the Electron-Phonon folder/Interaction file,



Adding V1 and V2 we get:



When we make the analytic continuation iνn­ → ω + iη, we obtain,



Since εRPA(q,ω)  εRPA(q,0) on the characteristic frequency scale of the phonons we see that Veff(q,ω) changes sign and becomes attractive for |ω| < ωq. Let’s fill in εRPA using the Thomas-Fermi approximation. Then we have, proceeding to the *small q* limit:



Filling in the 3D Coulomb potential, we’d get a q-independent result (well at least V(qTF) is q-independent):



Although the region of attractive interactions is only a small part of the frequency spectrum, none-the-less this effect turns out to have profound implications, which we’ll return to when we talk about superconductivity. We can imagine that we are restricted to small frequencies because the lattice is unable to screen the electron motion for large frequencies, due to its inertia.